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Level-rank duality of untwisted and twisted D-branes

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Abstract

Level-rank duality of untwisted and twisted D-branes of WZW models is explored. We derive the relation between D0-brane charges of level-rank dual untwisted D-branes of $\widehat{\mathrm{su}}(N)_K$ and $\widehat{\mathrm{sp}}(n)_k$, and of level-rank dual twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$. The analysis of level-rank duality of twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ is facilitated by their close relation to untwisted D-branes of $\widehat{\mathrm{sp}}(n)_k$. We also demonstrate level-rank duality of the spectrum of an open string stretched between untwisted or twisted D-branes in each of these cases.

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1 Introduction

D-branes on group manifolds have been the subject of much work, both from the algebraic and geometric points of view [1]–[26]. (For a review, see ref. [27]. See also ref. [28].) Algebraically, these D-branes correspond to the allowed boundary conditions for a Wess-Zumino-Witten (WZW) model on a surface with boundary [29].

Much can be learned about D-branes by studying their charges, which are classified by K-theory or, in the presence of a cohomologically nontrivial H-field background, twisted K-theory [30]. The charge group for D-branes on a simply-connected group manifold G with level K is given by the twisted K-group [10, 12, 31, 32, 33, 18]

$$K^*(G) = \oplus_{i=1}^{m} \mathbb{Z}_x\,, \qquad m = 2^{\mathrm{rank}\, G - 1} \tag{1.1}$$

where $\mathbb{Z}_x \equiv \mathbb{Z}/x\mathbb{Z}$ with x an integer depending on G and K. For $\widehat{\mathrm{su}}(N)_K$, for example, x is given by [10]

$$x_{N,K} \equiv \frac{N+K}{\gcd\{N+K, \mathrm{lcm}\{1,\ldots,N-1\}\}}\,. \tag{1.2}$$

One of the $\mathbb{Z}_x$ factors in the charge group corresponds to the charge of untwisted (symmetry-preserving) D-branes. For su(N) with $N > 2$, another of the $\mathbb{Z}_x$ factors corresponds to D-branes twisted by the charge-conjugation symmetry. For the D-branes corresponding to the remaining factors, see refs. [10, 12, 18].

WZW models with classical Lie groups possess an interesting property called level-rank duality: a relationship between various quantities in the $\widehat{\mathrm{su}}(N)_K$, $\widehat{\mathrm{so}}(N)_K$, or $\widehat{\mathrm{sp}}(n)_k$ model, and corresponding quantities in the level-rank dual $\widehat{\mathrm{su}}(K)_N$, $\widehat{\mathrm{so}}(K)_N$, or $\widehat{\mathrm{sp}}(k)_n$ model [34]–[37]. Implications of level-rank duality for boundary Kazama-Suzuki models were explored in ref. [24].

In ref. [38], we began the study of level-rank duality in boundary WZW theories, and in particular the level-rank duality of untwisted D-branes of $\widehat{\mathrm{su}}(N)_K$. In this paper, we extend this work to untwisted D-branes of the $\widehat{\mathrm{sp}}(n)_k$ WZW model, and to twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$, which are closely related to the untwisted D-branes of $\widehat{\mathrm{sp}}(n)_k$. We focus on two aspects of this duality: the relation between the D0-brane charges of level-rank dual D-branes, and the level-rank duality of the spectrum of an open string stretched between untwisted or twisted D-branes (*i.e.*, the coefficients of the open-string partition function). For untwisted D-branes, these coefficients are given by the fusion coefficients of the bulk WZW theory [29], so duality of the untwisted open-string partition function follows from the well-known level-rank duality of the fusion rules [34, 35, 36]. For twisted D-branes, the open-string partition function coefficients may be calculated in terms of the modular-transformation matrices of twisted affine Lie algebras [4, 14, 16]. In this paper, we show that the spectrum of an open string stretched between twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ is level-rank dual.

In section 2, we review some salient features of untwisted D-branes of WZW models. Section 3 describes the level-rank duality of the charges of untwisted D-branes of $\widehat{\mathrm{su}}(N)_K$ for all values of N and K (our results in ref. [38] were restricted to $N+K$ odd), and of the untwisted open-string partition function. Section 4 describes the level-rank duality of the charges of untwisted D-branes of $\widehat{\mathrm{sp}}(n)_k$, and of the untwisted open-string partition function.

Twisted D-branes of WZW models are reviewed in section 5, and section 6 is devoted to demonstrating the level-rank duality of the charges of twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$, and of the twisted open-string partition function. Concluding remarks constitute section 7.

2 Untwisted D-branes of WZW models

In this section, we review some salient features of Wess-Zumino-Witten models and their untwisted D-branes.

The WZW model, which describes strings propagating on a group manifold, is a rational conformal field theory whose chiral algebra (for both left- and right-movers) is the (untwisted) affine Lie algebra $\hat{g}_K$ at level K. The Dynkin diagram of $\hat{g}_K$ has one more node than that of the associated finite-dimensional Lie algebra g. Let $(m_0, m_1, \cdots, m_n)$ be the dual Coxeter labels of $\hat{g}_K$ (where $n = \mathrm{rank}\ g$) and $h^\vee = \sum_{i=0}^n m_i$ the dual Coxeter number of g. The Virasoro central charge of the WZW model is then $c = K \dim g / (K + h^\vee)$.

The building blocks of the WZW conformal field theory are integrable highest-weight representations V_λ of $\hat{g}_K$, that is, representations whose highest weight $\lambda \in P_+^K$ has non-negative Dynkin indices $(a_0, a_1, \cdots, a_n)$ satisfying

$$\sum_{i=0}^{n} m_i a_i = K \,. \tag{2.1}$$

With a slight abuse of notation, we also use λ to denote the highest weight of the irreducible representation of g with Dynkin indices $(a_1, \cdots, a_n)$, which spans the lowest-conformal-weight subspace of V_λ.

For $\widehat{\mathrm{su}}(n+1)_K = (A_n^{(1)})_K$ and $\widehat{\mathrm{sp}}(n)_K = (C_n^{(1)})_K$, the untwisted affine Lie algebras with which we will be principally concerned, we have $m_i = 1$ for $i = 0, \cdots, n$, and $h^\vee = n+1$. It is often useful to describe irreducible representations of g in terms of Young tableaux. For example, an irreducible representation of $\mathrm{su}(n+1)$ or $\mathrm{sp}(n)$ whose highest weight λ has Dynkin indices a_i corresponds to a Young tableau with n or fewer rows, with row lengths

$$\ell_i = \sum_{j=i}^{n} a_j \,, \qquad i = 1, \ldots, n \,. \tag{2.2}$$

Let $r(\lambda) = \sum_{i=1}^n \ell_i$ denote the number of boxes of the tableau. Representations λ corresponding to integrable highest-weight representations V_λ of $\widehat{\mathrm{su}}(n+1)_K$ or $\widehat{\mathrm{sp}}(n)_K$ have Young tableaux with K or fewer columns.

We will only consider WZW theories with a diagonal closed-string spectrum:

$$\mathcal{H}^{\mathrm{closed}} = \bigoplus_{\lambda \in P_+^K} V_\lambda \otimes \overline{V}_{\lambda^*} \tag{2.3}$$

where $\overline{V}$ represents right-moving states, and λ^* denotes the representation conjugate to λ. The partition function for this theory is

$$Z^{\mathrm{closed}}(\tau) = \sum_{\lambda \in P_+^K} |\chi_\lambda(\tau)|^2 \tag{2.4}$$

where

$$\chi_\lambda(\tau) = \mathrm{Tr}_{V_\lambda} q^{L_0 - c/24}\,, \qquad q = \mathrm{e}^{2\pi i \tau} \tag{2.5}$$

is the affine character of the integrable highest-weight representation V_λ. The affine characters transform linearly under the modular transformation $\tau \to -1/\tau$,

$$\chi_\lambda(-1/\tau) = \sum_{\mu \in P_+^K} S_{\mu\lambda}\, \chi_\mu(\tau)\,, \tag{2.6}$$

and the unitarity of S ensures the modular invariance of the partition function (2.4).

Next we turn to consider D-branes in the WZW model [1]-[26]. These D-branes may be studied algebraically in terms of the possible boundary conditions that can consistently be imposed on a WZW model with boundary. We consider boundary conditions that leave unbroken the $\hat{g}_K$ symmetry, as well as the conformal symmetry, of the theory, and we label the allowed boundary conditions (and therefore the D-branes) by α, β, $\cdots$. The partition function on a cylinder, with boundary conditions α and β on the two boundary components, is then given as a linear combination of affine characters of $\hat{g}_K$ [29]

$$Z^{\mathrm{open}}_{\alpha\beta}(\tau) = \sum_{\lambda \in P_+^K} n_{\beta\lambda}{}^{\alpha} \chi_\lambda(\tau)\,. \tag{2.7}$$

This describes the spectrum of an open string stretched between D-branes labelled by α and β.

In this section, we consider a special class of boundary conditions, called *untwisted* (or *symmetry-preserving*), that result from imposing the restriction

$$\left[J^a(z) - \overline{J}^a(\bar{z}) \right] \Big|_{z=\bar{z}} = 0 \tag{2.8}$$

on the currents of the affine Lie algebra on the boundary $z = \bar{z}$ of the open string worldsheet, which has been conformally transformed to the upper half plane. Open-closed string duality allows one to correlate the boundary conditions (2.8) of the boundary WZW model with coherent states $|B\rangle\rangle \in \mathcal{H}^{\mathrm{closed}}$ of the bulk WZW model satisfying

$$\left[J^a_m + \overline{J}^a_{-m} \right] |B\rangle\rangle = 0\,, \qquad m \in \mathbb{Z} \tag{2.9}$$

where J^a_m are the modes of the affine Lie algebra generators. Solutions of eq. (2.9) that belong to a single sector $V_\mu \otimes \overline{V}_{\mu^*}$ of the bulk WZW theory are known as Ishibashi states $|\mu\rangle\rangle_I$ [39], and are normalized such that

$${}_I\langle\langle \mu | q^H | \nu \rangle\rangle_I = \delta_{\mu\nu} \chi_\mu(\tau)\,, \qquad q = \mathrm{e}^{2\pi i \tau} \tag{2.10}$$

where $H = \frac{1}{2}\left(L_0 + \overline{L}_0 - \frac{1}{12} c \right)$ is the closed-string Hamiltonian. For the diagonal theory (2.3), Ishibashi states exist for all integrable highest-weight representations $\mu \in P_+^K$ of $\hat{g}_K$.

A coherent state $|B\rangle\rangle$ that corresponds to an allowed boundary condition must also satisfy additional (Cardy) conditions [29], among which are that the coefficients $n_{\beta\lambda}{}^{\alpha}$ in eq. (2.7) must be non-negative integers. Solutions to these conditions are labelled by integrable highest-weight representations $\lambda \in P_+^K$ of the untwisted affine Lie algebra $\hat{g}_K$, and

are known as (untwisted) Cardy states $|\lambda\rangle\!\rangle_C$. The Cardy states may be expressed as linear combinations of Ishibashi states

$$|\lambda\rangle\!\rangle_C = \sum_{\mu\in P_+^K} \frac{S_{\lambda\mu}}{\sqrt{S_{0\mu}}} |\mu\rangle\!\rangle_I \tag{2.11}$$

where $S_{\lambda\mu}$ is the modular transformation matrix given by eq. (2.6), and 0 denotes the identity representation. Untwisted D-branes of $\hat{g}_K$ correspond to $|\lambda\rangle\!\rangle_C$ and are therefore also labelled by $\lambda \in P_+^K$.

The partition function of open strings stretched between untwisted D-branes λ and μ

$$Z^{\rm open}_{\lambda\mu}(\tau) = \sum_{\nu\in P_+^K} n_{\mu\nu}{}^{\lambda} \chi_\nu(\tau) \tag{2.12}$$

may alternatively be calculated as the closed-string propagator between untwisted Cardy states [29]

$$Z^{\rm open}_{\lambda\mu}(\tau) = {}_C\langle\!\langle\lambda|\tilde{q}^H|\mu\rangle\!\rangle_C\,, \qquad \tilde{q} = {\rm e}^{2\pi i(-1/\tau)}\,. \tag{2.13}$$

Combining eqs. (2.13), (2.11), (2.10), (2.6), and the Verlinde formula [40], we find

$$Z^{\rm open}_{\lambda\mu}(\tau) = \sum_{\rho\in P_+^K} \frac{S^*_{\lambda\rho}S_{\mu\rho}}{S_{0\rho}} \chi_\rho(-1/\tau) = \sum_{\nu\in P_+^K}\sum_{\rho\in P_+^K} \frac{S_{\mu\rho}S_{\nu\rho}S^*_{\lambda\rho}}{S_{0\rho}} \chi_\nu(\tau) = \sum_{\nu\in P_+^K} N_{\mu\nu}{}^{\lambda}\chi_\nu(\tau)\,. \tag{2.14}$$

Hence, the coefficients $n_{\mu\nu}{}^{\lambda}$ in the open-string partition function (2.12) are simply given by the fusion coefficients $N_{\mu\nu}{}^{\lambda}$ of the bulk WZW model.

Finally, an untwisted D-brane labelled by $\lambda \in P_+^K$ can be considered a bound state of D0-branes [41, 5, 8, 9, 10, 12]. It possesses a conserved D0-brane charge Q_λ given by $(\dim\lambda)_g$, but the charge is only defined modulo some integer [9, 10, 12, 21]. For D-branes of $\widehat{\rm su}(N)_K$, for example, this integer is given by eq. (1.2), thus

$$Q_\lambda = (\dim\lambda)_{{\rm su}(N)} \mod x_{N,K} \qquad \text{for } \ \widehat{\rm su}(N)_K \tag{2.15}$$

is the charge of the untwisted D-brane labelled by λ.

3 Level-rank duality of untwisted D-branes of $\widehat{\rm su}(N)_K$

In ref. [38], the relation between the charges of untwisted D-branes of the $\widehat{\rm su}(N)_K$ model and those of the level-rank-dual $\widehat{\rm su}(K)_N$ model was ascertained for odd values of $N+K$. In this section, we extend these results to all values of N and K.

Since charges of $\widehat{\rm su}(N)_K$ D-branes are only defined modulo $x_{N,K}$, and those of $\widehat{\rm su}(K)_N$ D-branes modulo $x_{K,N}$, comparison of charges of level-rank-dual D-branes is only possible modulo $\gcd\{x_{N,K}, x_{K,N}\}$. Without loss of generality we will henceforth assume that $N \ge K$, in which case $\gcd\{x_{N,K}, x_{K,N}\} = x_{N,K}$.

Level-rank duality of untwisted D-brane charges

Given a Young tableau λ corresponding to an integrable highest-weight representation of $\widehat{\rm su}(N)_K$ (with $N-1$ or fewer rows, and K or fewer columns), its transpose $\tilde{\lambda}$ corresponds to

an integrable highest-weight representation of $\widehat{\mathrm{su}}(K)_N$. (The map between representations of $\widehat{\mathrm{su}}(N)_K$ and $\widehat{\mathrm{su}}(K)_N$ is not one-to-one, but the map between cominimal equivalence classes of representations is. These equivalence classes are generated by the simple-current symmetry σ of $\widehat{\mathrm{su}}(N)_K$, which takes λ into $\lambda' = \sigma(\lambda)$, where the Dynkin indices of λ' are $a_i' = a_{i-1}$ for $i = 1, \ldots, N-1$, and $a_0' = a_{N-1}$.)

For odd $N+K$, the relation between Q_λ, the charge of the untwisted $\widehat{\mathrm{su}}(N)_K$ D-brane labelled by λ, and $\tilde{Q}_{\tilde{\lambda}}$, the charge of the level-rank-dual $\widehat{\mathrm{su}}(K)_N$ D-brane labelled by $\tilde{\lambda}$, was shown to be [38]

$$\tilde{Q}_{\tilde{\lambda}} = (-1)^{r(\lambda)} Q_\lambda \mod x_{N,K}, \qquad \text{for } N+K \text{ odd}. \tag{3.1}$$

where $r(\lambda)$ is the number of boxes in the tableau λ. In this section, we show that for the case of even $N+K$, the charges obey

$$\tilde{Q}_{\tilde{\lambda}} = Q_\lambda \mod x_{N,K}, \qquad \text{for } N+K \text{ even (except for } N = K = 2^m). \tag{3.2}$$

In the remaining case, we conjecture the relation

$$\tilde{Q}_{\tilde{\lambda}} = \left\{ \begin{array}{ll} (-1)^{r(\lambda)/N} Q_\lambda \mod x_{N,N}, & \text{when } N \mid r(\lambda) \\ Q_\lambda \mod x_{N,N}, & \text{when } N \not| \, r(\lambda) \end{array} \right\} \quad \text{for } N = K = 2^m \tag{3.3}$$

for which we have numerical evidence, but (as of yet) no complete proof.

Proof of eq. (3.2): We proceed as in ref. [38] by writing the dimension of an arbitrary irreducible representation λ of $\mathrm{su}(N)$ (with row lengths ℓ_i and column lengths k_i) as the determinant of an $\ell_1 \times \ell_1$ matrix (eq. (A.6) of ref. [44])

$$(\dim \lambda)_{\mathrm{su}(N)} = \left| (\dim \Lambda_{k_i+j-i})_{\mathrm{su}(N)} \right|, \qquad i,j = 1, \ldots, \ell_1 \tag{3.4}$$

where Λ_s is the completely *antisymmetric* representation of $\mathrm{su}(N)$, whose Young tableau is $\left.\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right\}s$. The maximum value of s appearing in eq. (3.4) is $k_1 + \ell_1 - 1$, which is bounded by $N+K-2$ for integrable highest-weight representations of $\widehat{\mathrm{su}}(N)_K$. The representations Λ_0 and Λ_N both correspond to the identity representation, with dimension 1. For $1 \le s \le N-1$, Λ_s are the fundamental representations of $\mathrm{su}(N)$, with $(\dim \Lambda_s)_{\mathrm{su}(N)} = \binom{N}{s}$. We define $\dim \Lambda_s = 0$ for $s < 0$ and for $s > N$.

In ref. [38], we showed that

$$(\dim \Lambda_s)_{\mathrm{su}(N)} = \begin{cases} (-1)^s (\dim \tilde{\Lambda}_s)_{\mathrm{su}(K)} \mod x_{N,K}, & \text{for } s \le N+K-2, \text{ except } s = N \\ (-1)^{K-1} (\dim \tilde{\Lambda}_s)_{\mathrm{su}(K)} \mod x_{N,K}, & \text{for } s = N \end{cases} \tag{3.5}$$

where $\tilde{\Lambda}_s$ is the completely *symmetric* representation of $\mathrm{su}(K)$, whose Young tableau is $\underbrace{\square\square\square\square}_{s}$. (We define $\dim \tilde{\Lambda}_s = 0$ for $s < 0$.) When $N+K$ is odd, eq. (3.5) becomes simply $(\dim \Lambda_s)_{\mathrm{su}(N)} = (-1)^s (\dim \tilde{\Lambda}_s)_{\mathrm{su}(K)} \mod x_{N,K}$ for all $s \le N+K-2$. This was used in ref. [38] to yield eq. (3.1).

Now we turn to the case of even $N+K$, first considering $N > K$. In eq. (1.2), the factor $\mathrm{lcm}\{1, \ldots, N-1\}$ then contains $(N+K)/2$, so $x_{N,K}$ is at most 2. It is easy to see that

$x_{N,K} = 2$ if $N+K = 2^m$, and $x_{N,K} = 1$ otherwise. For $x_{N,K} \le 2$, the minus signs in eq. (3.5) are irrelevant (since $n = -n \bmod 2$), so we may simply write

$$(\dim \Lambda_s)_{\mathrm{su}(N)} = (\dim \tilde{\Lambda}_s)_{\mathrm{su}(K)} \mod x_{N,K}, \quad \text{for } s \le N+K-2, \text{ with } N+K \text{ even and } N > K\,. \tag{3.6}$$

We will use this below.

Next we consider $N = K$. We begin by observing that if N is a power of a prime p, then $x_{N,N} = 4$ if $p = 2$, and $x_{N,N} = p$ if $p > 2$. If N contains more than one prime factor, then $x_{N,N} = 1$. In the latter case, eq. (3.2) is trivially satisfied, so we need only consider $N = K = p^m$, where p is prime. Let us obtain the relation between $(\dim \Lambda_s)_{\mathrm{su}(p^m)}$ and $(\dim \tilde{\Lambda}_s)_{\mathrm{su}(p^m)}$ by considering three separate cases:

- $0 \le s \le N-1$:

  By examining the factors of p (prime) in the numerator and denominator of $(\dim \Lambda_s)_{\mathrm{su}(p^m)} = \binom{p^m}{s}$, one can establish that if p^{l-1} divides s but p^l does not (for any $l \le m$), then p^{m-l+1} divides $\binom{p^m}{s}$. Thus $(\dim \Lambda_s)_{\mathrm{su}(p^m)} = 0 \bmod p$ for $1 \le s \le N-1$. Combining this with eq. (3.5), we have

  $$(\dim \Lambda_s)_{\mathrm{su}(p^m)} = (\dim \tilde{\Lambda}_s)_{\mathrm{su}(p^m)} \mod x_{N,N}, \qquad \text{for } 1 \le s \le N-1\,. \tag{3.7}$$

  This is trivially extended to $s = 0$.

- $s < 0$, or $N+1 \le s \le 2N-2$:

  In this case,

  $$(\dim \Lambda_s)_{\mathrm{su}(p^m)} = (\dim \tilde{\Lambda}_s)_{\mathrm{su}(p^m)} \mod x_{N,N}, \qquad \text{for } s < 0, \text{ or } N+1 \le s \le 2N-2\,, \tag{3.8}$$

  is valid because the l. h. s. vanishes, and so, by eq. (3.5), the r. h. s. either vanishes or is a multiple of $x_{N,N}$.

- $s = N$:

  The remaining case yields [38]

  $$(\dim \Lambda_N)_{\mathrm{su}(p^m)} = (-1)^{N-1} (\dim \tilde{\Lambda}_N)_{\mathrm{su}(p^m)} \mod x_{N,N} \tag{3.9}$$

  which is in accord with the other cases when p is a prime other than 2.

We combine these results with eq. (3.6) to write

$$(\dim \Lambda_s)_{\mathrm{su}(N)} = (\dim \tilde{\Lambda}_s)_{\mathrm{su}(K)} \qquad \mod x_{N,K}, \quad \text{for } s \le N+K-2, \quad \text{for } N+K \text{ even (except } N = K = 2^m). \tag{3.10}$$

Inserting this in eq. (3.4), we find

$$(\dim \lambda)_{\mathrm{su}(N)} = \left| (\dim \tilde{\Lambda}_{k_i+j-i})_{\mathrm{su}(K)} \right| \mod x_{N,K}, \quad \text{for } N+K \text{ even (except } N = K = 2^m). \tag{3.11}$$

By an alternative formula for the dimension of a representation (eq. (A.5) of ref. [44]), the r.h.s. is the dimension of a representation of su(K) with row lengths k_i and column lengths ℓ_i,[3] that is, the transpose representation $\tilde{\lambda}$, hence

$$(\dim\lambda)_{\mathrm{su}(N)} = (\dim\tilde{\lambda})_{\mathrm{su}(K)} \mod x_{N,K}, \quad \text{for } N+K \text{ even (except } N=K=2^m). \tag{3.12}$$

from which eq. (3.2) follows.[4]

Level-rank duality of the untwisted open string spectrum

In ref. [35, 36], it was shown that the fusion coefficients $N_{\mu\nu}{}^{\lambda}$ of the bulk $\widehat{\mathrm{su}}(N)_K$ WZW model are related to those of the $\widehat{\mathrm{su}}(K)_N$ WZW model, denoted by $\tilde{N}$, by

$$N_{\mu\nu}{}^{\lambda} = \tilde{N}_{\tilde{\mu}\tilde{\nu}}{}^{\sigma^{\Delta}(\tilde{\lambda})} = \tilde{N}_{\tilde{\mu}\sigma^{-\Delta}(\tilde{\nu})}{}^{\tilde{\lambda}} \tag{3.13}$$

where $\Delta = [r(\mu)+r(\nu)-r(\lambda)]/N$.

Since by eq. (2.14) the fusion coefficients $N_{\mu\nu}{}^{\lambda}$ are equal to the coefficients $n_{\mu\nu}{}^{\lambda}$ of the open-string partition function (2.12), it follows that if the spectrum of an $\widehat{\mathrm{su}}(N)_K$ open string stretched between untwisted D-branes λ and μ contains $n_{\mu\nu}{}^{\lambda}$ copies of the highest-weight representation V_ν of $\widehat{\mathrm{su}}(N)_K$, then the spectrum of an $\widehat{\mathrm{su}}(K)_N$ open string stretched between untwisted D-branes $\tilde{\lambda}$ and $\tilde{\mu}$ contains an equal number of copies of the highest-weight representation $V_{\sigma^{-\Delta}(\tilde{\nu})}$ of $\widehat{\mathrm{su}}(K)_N$.

4 Level-rank duality of untwisted D-branes of $\widehat{\mathrm{sp}}(n)_k$

In this section, we examine the relation between untwisted D-branes of the $\widehat{\mathrm{sp}}(n)_k$ model and those of the level-rank-dual $\widehat{\mathrm{sp}}(k)_n$ model.

Untwisted D-branes of $\widehat{\mathrm{sp}}(n)_k$ are labelled by integrable highest-weight representations V_λ of $\widehat{\mathrm{sp}}(n)_k = (C_n^{(1)})_k$. The D0-brane charge of D-branes of $\widehat{\mathrm{sp}}(n)_k$ are defined modulo the integer [21, 17]

$$\begin{aligned}
x &= \frac{n+k+1}{\gcd\{n+k+1, \mathrm{lcm}\{1,2,3,\ldots,n,1,3,5,\ldots,2n-1\}\}} \\
&= \frac{n+k+1}{\gcd\{n+k+1, \frac{1}{2}\mathrm{lcm}\{1,2,\ldots,2n\}\}} \\
&= \frac{2(n+k+1)}{\gcd\{2(n+k+1), \mathrm{lcm}\{1,2,\ldots,2n\}\}} \\
&= x_{2n+1,2k+1}
\end{aligned} \tag{4.1}$$

[3] If λ has $\ell_1 = K$, then the transpose $\tilde{\lambda}$ contains leading columns of K boxes. In that case, one can apply the formula [12] $Q_{\sigma(\lambda)} = (-1)^{N-1}Q_\lambda \mod x_{N,K}$ several times to relate λ to a tableau with no rows of length K before using eq. (3.4). The minus sign is irrelevant when $x_{N,K} \le 2$, and vanishes when N is an odd prime.

[4] Since minus signs are irrelevant when $x_{N,K} \le 2$, eq. (3.1) actually holds for all $N \ne K$, not just odd $N+K$. Equation (3.1) is not valid, however, when $N = K$. This is most easily seen by considering representations of $\widehat{\mathrm{su}}(N)_N$ whose tableaux are invariant under transposition, and whose dimensions are not multiples of x, such as the adjoint of $\widehat{\mathrm{su}}(3)_3$.

where $x_{2n+1,2k+1}$ is given by eq. (1.2). That is,

$$Q_\lambda = (\dim\lambda)_{\mathrm{sp}(n)} \mod x_{2n+1,2k+1} \qquad \text{for} \quad \widehat{\mathrm{sp}}(n)_k \tag{4.2}$$

is the charge of the untwisted $\widehat{\mathrm{sp}}(n)_k$ D-brane labelled by λ, where $(\dim\lambda)_{\mathrm{sp}(n)}$ is the dimension of the $\mathrm{sp}(n)$ representation λ. As we showed in the previous section, for $n \neq k$, we have $x_{2n+1,2k+1} = 2$ if $n+k+1 = 2^m$, and $x_{2n+1,2k+1} = 1$ otherwise. For $n = k$, we have $x_{2n+1,2n+1} = p$ if $2n+1 = p^m$, and $x_{2n+1,2n+1} = 1$ if $2n+1$ contains more than one prime factor.

Since charges of $\widehat{\mathrm{sp}}(n)_k$ D-branes are only defined modulo $x_{2n+1,2k+1}$, and those of $\widehat{\mathrm{sp}}(k)_n$ D-branes modulo $x_{2k+1,2n+1}$, comparison of charges of level-rank-dual D-branes is only possible modulo $\gcd\{x_{2n+1,2k+1}, x_{2k+1,2n+1}\}$. Without loss of generality we henceforth assume that $n \geq k$, in which case $\gcd\{x_{2n+1,2k+1}, x_{2k+1,2n+1}\} = x_{2n+1,2k+1}$.

Level-rank duality of untwisted D-brane charges

Given a Young tableau λ corresponding to an integrable highest-weight representation of $\widehat{\mathrm{sp}}(n)_k$ (with n or fewer rows and k or fewer columns), its transpose $\tilde\lambda$ corresponds to an integrable highest-weight representation of $\widehat{\mathrm{sp}}(k)_n$. The mapping between representations is one-to-one, in contrast to the case of $\widehat{\mathrm{su}}(N)_K$.

We will show that the relation between Q_λ, the charge of the $\widehat{\mathrm{sp}}(n)_k$ D-brane labelled by λ, and $\tilde Q_{\tilde\lambda}$, the charge of the level-rank-dual $\widehat{\mathrm{sp}}(k)_n$ D-brane labelled by $\tilde\lambda$, is given by

$$\tilde Q_{\tilde\lambda} = Q_\lambda \mod x_{2n+1,2k+1}\,. \tag{4.3}$$

The relation (4.3) is nontrivial only when $x_{2n+1,2k+1} > 1$, that is, when $n \neq k$ with $n+k+1 = 2^m$, or when $n = k$ with $2n+1 = p^m$.

Proof of eq. (4.3): We may write the dimension of an arbitrary irreducible representation λ of $\mathrm{sp}(n)$ as the determinant of an $\ell_1 \times \ell_1$ matrix (Prop. (A.44) of ref. [44]; see also ref. [37])

$$(\dim\lambda)_{\mathrm{sp}(n)} = \begin{vmatrix} \chi_{k_1} & (\chi_{k_1+1} + \chi_{k_1-1}) & \cdots & (\chi_{k_1+\ell_1-1} + \chi_{k_1-\ell_1+1}) \\ \vdots & \vdots & \vdots & \vdots \\ \chi_{k_i-i+1} & (\chi_{k_i-i+2} + \chi_{k_i-i}) & \cdots & (\chi_{k_1+\ell_1-i} + \chi_{k_1-\ell_1-i+2}) \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}\,, \qquad i,j = 1,\ldots,\ell_1 \tag{4.4}$$

where $\chi_s = (\dim\Lambda_s)_{\mathrm{sp}(n)}$, with Λ_s the completely *antisymmetric* representation of $\mathrm{sp}(n)$, whose Young tableau is $\left.\begin{smallmatrix}\square\\ \vdots\\ \square\end{smallmatrix}\right\}s$. The maximum value of s appearing in eq. (4.4) is $k_1 + \ell_1 - 1$, which is bounded by $n+k-1$ for integrable highest-weight representations of $\widehat{\mathrm{sp}}(n)_k$. The representation Λ_0 corresponds to the identity representation with dimension 1. For $1 \leq s \leq n$, Λ_s are the fundamental representations of $\mathrm{sp}(n)$. (We define $(\dim\Lambda_s)_{\mathrm{sp}(n)} = 0$ for $s < 0$ and for $s > n$.) Also, let $\tilde\Lambda_s$ be the completely *symmetric* representation of $\mathrm{sp}(k)$, whose Young tableau is $\underbrace{\square\square\cdots\square}_{s}$. (We define $(\dim\tilde\Lambda_s)_{\mathrm{sp}(k)} = 0$ for $s < 0$.)

Next, we may use the branching rules $(\Lambda_s)_{\mathrm{su}(2n+1)} = \oplus_{t=0}^{s}(\Lambda_t)_{\mathrm{sp}(n)}$ (for $s \leq n$) and $(\tilde\Lambda_s)_{\mathrm{su}(2n+1)} = \oplus_{t=0}^{s}(\tilde\Lambda_t)_{\mathrm{sp}(n)}$ of $\mathrm{su}(2n+1) \supset \mathrm{sp}(n)$ to relate the dimensions of the fundamental representations of $\mathrm{sp}(n)$ to those of the fundamental representations of $\mathrm{su}(2n+1)$:

$$(\dim\Lambda_s)_{\mathrm{sp}(n)} = (\dim\Lambda_s)_{\mathrm{su}(2n+1)} - (\dim\Lambda_{s-1})_{\mathrm{su}(2n+1)}\,,$$

$$(\dim \tilde{\Lambda}_s)_{\mathrm{sp}(k)} \quad = \quad (\dim \tilde{\Lambda}_s)_{\mathrm{su}(2k+1)} - (\dim \tilde{\Lambda}_{s-1})_{\mathrm{su}(2k+1)} \,. \tag{4.5}$$

Using this together with eq. (3.10), we have

$$(\dim \Lambda_s)_{\mathrm{sp}(n)} = (\dim \tilde{\Lambda}_s)_{\mathrm{sp}(k)} \quad \mathrm{mod}\ x_{2n+1,2k+1}, \qquad \text{for } s \le 2n+2k\,. \tag{4.6}$$

We use this in eq. (4.4) to obtain

$$(\dim \lambda)_{\mathrm{sp}(n)} = \begin{vmatrix} \tilde{\chi}_{k_1} & (\tilde{\chi}_{k_1+1} + \tilde{\chi}_{k_1-1}) & \cdots & (\tilde{\chi}_{k_1+\ell_1-1} + \tilde{\chi}_{k_1-\ell_1+1}) \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{\chi}_{k_i-i+1} & (\tilde{\chi}_{k_i-i+2} + \tilde{\chi}_{k_i-i}) & \cdots & (\tilde{\chi}_{k_1+\ell_1-i} + \tilde{\chi}_{k_1-\ell_1-i+2}) \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix} \quad \mathrm{mod}\ x_{2n+1,2k+1} \tag{4.7}$$

where $\tilde{\chi}_s = (\dim \tilde{\Lambda}_s)_{\mathrm{sp}(k)}$. By an alternative formula for the dimension of a representation (Prop. (A.50) of ref. [44]), the r.h.s. is the dimension of a representation of $\mathrm{sp}(k)$ with row lengths k_i and column lengths ℓ_i, that is, the transpose representation $\tilde{\lambda}$, hence

$$(\dim \lambda)_{\mathrm{sp}(n)} = (\dim \tilde{\lambda})_{\mathrm{sp}(k)} \quad \mathrm{mod}\ x_{2n+1,2k+1}, \tag{4.8}$$

from which eq. (4.3) follows. *QED.*

Level-rank duality of the untwisted open string spectrum

In ref. [36], it was shown that the fusion coefficients $N_{\mu\nu}{}^{\lambda}$ of the bulk $\widehat{\mathrm{sp}}(n)_k$ WZW model are related to those of the $\widehat{\mathrm{sp}}(k)_n$ WZW model by

$$N_{\mu\nu}{}^{\lambda} = \tilde{N}_{\tilde{\mu}\tilde{\nu}}{}^{\tilde{\lambda}} \,. \tag{4.9}$$

Since the fusion coefficients $N_{\mu\nu}{}^{\lambda}$ are equal to the coefficients $n_{\mu\nu}{}^{\lambda}$ of the open-string partition function, it follows that if the spectrum of an $\widehat{\mathrm{sp}}(n)_k$ open string stretched between untwisted D-branes λ and μ contains $n_{\mu\nu}{}^{\lambda}$ copies of the highest-weight representation V_ν of $\widehat{\mathrm{sp}}(n)_k$, then the spectrum of an $\widehat{\mathrm{sp}}(k)_n$ open string stretched between untwisted D-branes $\tilde{\lambda}$ and $\tilde{\mu}$ contains an equal number of copies of the highest-weight representation $V_{\tilde{\nu}}$ of $\widehat{\mathrm{sp}}(k)_n$.

5 Twisted D-branes of WZW models

In this section we review some aspects of twisted D-branes of the WZW model, drawing on refs. [2, 3, 4, 16]. As in section 2, these D-branes correspond to possible boundary conditions that can imposed on a boundary WZW model.

A boundary condition more general than eq. (2.8) that still preserves the $\hat{g}_K$ symmetry of the boundary WZW model is

$$\left[J^a(z) - \omega \overline{J}^a(\bar{z}) \right] \Big|_{z=\bar{z}} = 0\,, \tag{5.1}$$

where ω is an automorphism of the Lie algebra g. The boundary conditions (5.1) correspond to coherent states $|B\rangle\rangle^{\omega} \in \mathcal{H}^{\mathrm{closed}}$ of the bulk WZW model that satisfy

$$\left[J^a_m + \omega \overline{J}^a_{-m} \right] |B\rangle\rangle^{\omega} = 0\,, \qquad m \in \mathbb{Z}\,. \tag{5.2}$$

The ω-twisted Ishibashi states $|\mu\rangle\rangle_I^\omega$ are solutions of eq. (5.2) that belong to a single sector $V_\mu \otimes \overline{V}_{\omega(\mu)^*}$ of the bulk WZW theory, and whose normalization is given by

$$ {}_I^\omega\langle\langle\mu|q^H|\nu\rangle\rangle_I^\omega = \delta_{\mu\nu}\chi_\mu(\tau)\,, \qquad q = \mathrm{e}^{2\pi i\tau}\,. \tag{5.3} $$

Since we are considering the diagonal closed-string theory (2.3), these states only exist when $\mu = \omega(\mu)$, so the ω-twisted Ishibashi states are labelled by $\mu \in \mathcal{E}^\omega$, where $\mathcal{E}^\omega \subset P_+^K$ are the integrable highest-weight representations of $\hat{g}_K$ that satisfy $\omega(\mu) = \mu$. Equivalently, μ corresponds to a highest-weight representation, which we denote by $\pi(\mu)$, of $\breve{g}$, the orbit Lie algebra [42] associated with $\hat{g}_K$.

Solutions of eq. (5.2) that also satisfy the Cardy conditions are denoted ω-twisted Cardy states $|\alpha\rangle\rangle_C^\omega$, where the labels α take values in some set $\mathcal{B}^\omega$. The ω-twisted Cardy states may be expressed as linear combinations of ω-twisted Ishibashi states

$$ |\alpha\rangle\rangle_C^\omega = \sum_{\mu\in\mathcal{E}^\omega} \frac{\psi_{\alpha\pi(\mu)}}{\sqrt{S_{0\mu}}}|\mu\rangle\rangle_I^\omega \tag{5.4} $$

where $\psi_{\alpha\pi(\mu)}$ are some as-yet-undetermined coefficients. The ω-twisted D-branes of $\hat{g}_K$ correspond to $|\alpha\rangle\rangle_C^\omega$ and are therefore also labelled by $\alpha \in \mathcal{B}^\omega$. These states (apparently) correspond [4] to integrable highest-weight representations of the ω-twisted affine Lie algebra $\hat{g}_K^\omega$ (but see ref. [19]).

The partition function of open strings stretched between ω-twisted D-branes α and β

$$ Z_{\alpha\beta}^{\mathrm{open}}(\tau) = \sum_{\lambda\in P_+^K} n_{\beta\lambda}{}^\alpha \chi_\lambda(\tau) \tag{5.5} $$

may alternatively be calculated as the closed-string propagator between ω-twisted Cardy states

$$ Z_{\alpha\beta}^{\mathrm{open}}(\tau) = {}_C^\omega\langle\langle\alpha|\tilde{q}^H|\beta\rangle\rangle_C^\omega\,, \qquad \tilde{q} = \mathrm{e}^{2\pi i(-1/\tau)}\,. \tag{5.6} $$

Combining eqs. (5.6), (5.4), (5.3), and (2.6), we find

$$ Z_{\alpha\beta}^{\mathrm{open}}(\tau) = \sum_{\rho\in\mathcal{E}^\omega} \frac{\psi^*_{\alpha\pi(\rho)}\psi_{\beta\pi(\rho)}}{S_{0\rho}}\chi_\rho(-1/\tau) = \sum_{\lambda\in P_+^K}\sum_{\rho\in\mathcal{E}^\omega} \frac{\psi^*_{\alpha\pi(\rho)}S_{\lambda\rho}\psi_{\beta\pi(\rho)}}{S_{0\rho}}\chi_\lambda(\tau)\,. \tag{5.7} $$

Hence, the coefficients of the open-string partition function (5.5) are given by

$$ n_{\beta\lambda}{}^\alpha = \sum_{\rho\in\mathcal{E}^\omega} \frac{\psi^*_{\alpha\pi(\rho)}S_{\lambda\rho}\psi_{\beta\pi(\rho)}}{S_{0\rho}}\,. \tag{5.8} $$

Finally, the coefficients $\psi_{\alpha\pi(\rho)}$ relating the ω-twisted Cardy states and ω-twisted Ishibashi states may be identified [4] with the modular transformation matrices of characters of twisted affine Lie algebras [46], as may be seen, for example, by examining the partition function of an open string stretched between an ω-twisted and an untwisted D-brane [14, 16].

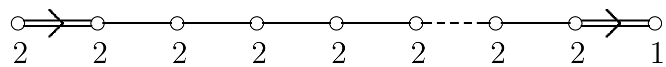
6 Level-rank duality of twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$

The finite Lie algebra $\mathrm{su}(N)$ possesses an order-two automorphism ω_c arising from the invariance of its Dynkin diagram under reflection. This automorphism maps the Dynkin indices of an irreducible representation $a_i \to a_{N-i}$, and corresponds to charge conjugation of the representation. This automorphism lifts to an automorphism of the affine Lie algebra $\widehat{\mathrm{su}}(N)_K$, leaving the zero$^{\mathrm{th}}$ node of the extended Dynkin diagram invariant, and gives rise to a class of ω_c-twisted D-branes of the $\widehat{\mathrm{su}}(N)_K$ WZW model (for $N > 2$). Since the details of the ω_c-twisted D-branes differ significantly between even and odd N, and we will restrict our attention to the ω_c-twisted D-branes of the $\widehat{\mathrm{su}}(2n+1)_{2k+1} = (A_{2n}^{(1)})_{2k+1}$ WZW model.

First, recall that the ω_c-twisted Ishibashi states $|\mu\rangle\!\rangle_I^{\omega_c}$ are labelled by self-conjugate integrable highest-weight representations $\mu \in \mathcal{E}^\omega$ of $(A_{2n}^{(1)})_{2k+1}$. Equation (2.1) implies that the Dynkin indices $(a_0, a_1, a_2, \cdots, a_{n-1}, a_n, a_n, a_{n-1}, \cdots, a_1)$ of μ satisfy

$$a_0 + 2(a_1 + \cdots + a_n) = 2k+1\,. \tag{6.1}$$

In ref. [42], it was shown that the self-conjugate highest-weight representations of $(A_{2n}^{(1)})_{2k+1}$ are in one-to-one correspondence with integrable highest weight representations of the associated orbit Lie algebra $\breve{g} = (A_{2n}^{(2)})_{2k+1}$, whose Dynkin diagram is



with the integers indicating the dual Coxeter label m_i of each node. The representation $\mu \in \mathcal{E}^\omega$ corresponds to the $(A_{2n}^{(2)})_{2k+1}$ representation $\pi(\mu)$ with Dynkin indices $(a_0, a_1, \cdots, a_n)$. Consistency with eq. (6.1) requires that the dual Coxeter labels are $(m_0, m_1, \cdots, m_n) = (1, 2, 2, \cdots, 2)$, and hence we must choose as the zero$^{\mathrm{th}}$ node the *right-most* node of the Dynkin diagram above. The finite part of the orbit Lie algebra $\breve{g}$, obtained by omitting the zero$^{\mathrm{th}}$ node, is thus C_n. (Note that C_n is the orbit Lie algebra of the finite Lie algebra A_{2n} [42].)

Observe that, by eq. (6.1), a_0 must be odd, and that the representation $\pi(\mu)$ of the orbit algebra $\breve{g}$ is in one-to-one correspondence [42, 15, 16] with the integrable highest-weight representation $\pi(\mu)'$ of the untwisted affine Lie algebra $(C_n^{(1)})_k$ with Dynkin indices $(a_0', a_1', \cdots, a_n')$, where $a_0' = \frac{1}{2}(a_0 - 1)$ and $a_i' = a_i$ for $i = 1, \cdots, n$.

Next, the ω_c-twisted Cardy states $|\alpha\rangle\!\rangle_C^{\omega_c}$ (and therefore the ω_c-twisted D-branes) of the $(A_{2n}^{(1)})_{2k+1}$ WZW model are (apparently) labelled [4] by the integrable highest-weight representations $\alpha \in \mathcal{B}^{\omega_c}$ of the twisted Lie algebra $\hat{g}_{2k+1}^{\omega_c} = (A_{2n}^{(2)})_{2k+1}$ (but see ref. [19]). We adopt the same convention as above for the labelling of the nodes of the Dynkin diagram (consistent with refs. [46, 16] but differing from refs. [43, 45]). Thus, the Dynkin indices $(a_0, a_1, \cdots, a_n)$ of the highest weights α must also satisfy eq. (6.1), and the ω_c-twisted D-branes are therefore characterized [16, 19] by the irreducible representations of $C_n = \mathrm{sp}(n)$ with Dynkin indices $(a_1, \cdots, a_n)$ (also denoted, with a slight abuse of notation, by α). The charge of the ω_c-twisted D-brane of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ labelled by α is given by [17]

$$Q_\alpha^{\omega_c} = (\dim \alpha)_{\mathrm{sp}(n)} \mod x_{2n+1,2k+1} \qquad \text{for } \widehat{\mathrm{su}}(2n+1)_{2k+1}\,. \tag{6.2}$$

The periodicity of the charge is the same as that of all D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$.

Observe also that the ω_c-twisted D-branes $\alpha \in \mathcal{B}^{\omega_c}$ are in one-to-one correspondence with integrable highest-weight representations α' of the untwisted affine Lie algebra $(C_n^{(1)})_k$ with Dynkin indices $(a'_0, a'_1, \cdots, a'_n)$, where $a'_0 = \frac{1}{2}(a_0 - 1)$ and $a'_i = a_i$ for $i = 1, \cdots, n$. That is, both the ω_c-twisted Ishibashi states and the ω_c-twisted Cardy states of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ are classified by integrable representations of $\widehat{\mathrm{sp}}(n)_k$.

Recall from eq. (5.8) that the coefficients of the partition function of open strings stretched between ω_c-twisted D-branes α and β are given by

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\rho \in \mathcal{E}^{\omega}} \frac{\psi^*_{\alpha\pi(\rho)} S_{\lambda\rho} \psi_{\beta\pi(\rho)}}{S_{0\rho}} \tag{6.3}$$

where α, $\beta \in \mathcal{B}^{\omega_c}$, $\lambda \in P_+^K$, and $\pi(\rho)$ is the representation of the orbit Lie algebra $(A_{2n}^{(2)})_{2k+1}$ that corresponds to the self-conjugate representation ρ of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$. The coefficients $\psi_{\alpha\pi(\rho)}$ are given [4, 14, 16] by the modular transformation matrix of the characters of $(A_{2n}^{(2)})_{2k+1}$. These in turn may be identified [42, 15, 16] with $S'_{\alpha'\pi(\rho)'}$, the modular transformation matrix of $(C_n^{(1)})_k = \widehat{\mathrm{sp}}(n)_k$, so

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\rho \in \mathcal{E}^{\omega}} \frac{S'^*_{\alpha'\pi(\rho)'} S_{\lambda\rho} S'_{\beta'\pi(\rho)'}}{S_{0\rho}} \,. \tag{6.4}$$

We will use this below to demonstrate level-rank duality of $n_{\beta\lambda}{}^{\alpha}$.

Level-rank duality of twisted D-brane charges

It is now straightforward to show the equality of charges of level-rank-dual ω_c-twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$. As seen above, the ω_c-twisted $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ D-brane labelled by α is in one-to-one correspondence with an integrable highest-weight representation α' of $\widehat{\mathrm{sp}}(n)_k$, and has the same charge (6.2) as the untwisted $\widehat{\mathrm{sp}}(n)_k$ D-brane labelled by α' (4.2), including periodicity. The integrable highest-weight representation α' of $\widehat{\mathrm{sp}}(n)_k$ is level-rank-dual to the integrable highest-weight representation $\tilde{\alpha}'$ of $\widehat{\mathrm{sp}}(k)_n$ obtained by transposing the Young tableau corresponding to α', and the charges of the corresponding untwisted D-branes obey

$$(\dim \alpha')_{\mathrm{sp}(n)} = (\dim \tilde{\alpha}')_{\mathrm{sp}(k)} \mod x_{2n+1,2k+1}, \tag{6.5}$$

as shown in sec. 4. Therefore the ω_c-twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ are in one-to-one correspondence with the ω_c-twisted D-branes of $\widehat{\mathrm{su}}(2k+1)_{2n+1}$, and the charges of level-rank-dual ω_c-twisted D-branes obey

$$Q_{\alpha}^{\omega_c} = \tilde{Q}_{\tilde{\alpha}}^{\omega_c} \mod x_{2n+1,2k+1} \tag{6.6}$$

where the map between ω_c-twisted D-branes is given by transposition of the associated $\widehat{\mathrm{sp}}(n)_k$ tableaux.

Level-rank duality of the twisted open string spectrum

The coefficients of the partition function of open strings stretched between ω_c-twisted D-branes α and β are real numbers so we may write (6.4) as

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\rho \in \mathcal{E}^{\omega}} \frac{S'_{\alpha'\pi(\rho)'} S^*_{\lambda\rho} S'^*_{\beta'\pi(\rho)'}}{S^*_{0\rho}} \,. \tag{6.7}$$

Under level-rank duality, the $\widehat{\mathrm{su}}(N)_K$ modular transformation matrices transform as [35, 36]

$$S_{\lambda\mu} = \sqrt{\frac{K}{N}} \mathrm{e}^{-2\pi i r(\lambda) r(\mu)/NK} \tilde{S}^*_{\tilde{\lambda}\tilde{\mu}} \tag{6.8}$$

and the (real) $\widehat{\mathrm{sp}}(n)_k$ modular transformation matrices transform as [36]

$$S'_{\alpha'\beta'} = \tilde{S}'_{\tilde{\alpha}'\tilde{\beta}'} = \tilde{S}'^*_{\tilde{\alpha}'\tilde{\beta}'} \tag{6.9}$$

where $\tilde{S}$ and $\tilde{S}'$ denote the $\widehat{\mathrm{su}}(K)_N$ and $\widehat{\mathrm{sp}}(k)_n$ modular transformation matrices respectively, $\tilde{\mu}$ is the transpose of the Young tableau corresponding to the $\widehat{\mathrm{su}}(N)_K$ representation μ, and $\tilde{\alpha}'$ is the transpose of the Young tableau corresponding to the $\widehat{\mathrm{sp}}(n)_k$ representation α'. These imply

$$\begin{aligned} n_{\beta\lambda}{}^{\alpha} &= \sum_{\rho\in\mathcal{E}^\omega} \frac{\tilde{S}'^*_{\tilde{\alpha}'\widetilde{\pi(\rho)'}} \tilde{S}_{\tilde{\lambda}\tilde{\rho}} \tilde{S}'_{\tilde{\beta}'\widetilde{\pi(\rho)'}}}{\tilde{S}_{0\tilde{\rho}}} \mathrm{e}^{2\pi i r(\lambda) r(\rho)/(2n+1)(2k+1)} \\ &= \sum_{\rho\in\mathcal{E}^\omega} \frac{\tilde{\psi}^*_{\tilde{\alpha}\widetilde{\pi(\rho)}} \tilde{S}_{\tilde{\lambda}\tilde{\rho}} \tilde{\psi}_{\tilde{\beta}\widetilde{\pi(\rho)}}}{\tilde{S}_{0\tilde{\rho}}} \mathrm{e}^{2\pi i r(\lambda) r(\rho)/(2n+1)(2k+1)} . \end{aligned} \tag{6.10}$$

Let $\hat{\rho}$ be the self-conjugate $\widehat{\mathrm{su}}(2k+1)_{2n+1}$ representation that maps to the $\widehat{\mathrm{sp}}(k)_n$ representation $\widetilde{\pi(\rho)}'$, which is the transpose of the $\widehat{\mathrm{sp}}(n)_k$ representation $\pi(\rho)'$. In other words, the representation $\pi(\hat{\rho})$ of the orbit algebra is identified with $\widetilde{\pi(\rho)}$. Now $\hat{\rho}$ is not equal to $\tilde{\rho}$ (the transpose of ρ), which is generally not a self-conjugate representation, but they are in the same cominimal equivalence class,

$$\tilde{\rho} = \sigma^{r(\rho)/(2n+1)}(\hat{\rho}) , \tag{6.11}$$

which we prove at the end of this section. Equation (6.11) implies that [35, 36]

$$\tilde{S}_{\tilde{\lambda}\tilde{\rho}} = \mathrm{e}^{-2\pi i r(\lambda) r(\rho)/(2n+1)(2k+1)} \tilde{S}_{\tilde{\lambda}\hat{\rho}} \tag{6.12}$$

so that eq. (6.10) becomes

$$n_{\beta\lambda}{}^{\alpha} = \sum_{\hat{\rho}} \frac{\tilde{\psi}^*_{\tilde{\alpha}\pi(\hat{\rho})} \tilde{S}_{\tilde{\lambda}\hat{\rho}} \tilde{\psi}_{\tilde{\beta}\pi(\hat{\rho})}}{\tilde{S}_{0\hat{\rho}}} = \tilde{n}_{\tilde{\beta}\tilde{\lambda}}{}^{\tilde{\alpha}} , \tag{6.13}$$

proving the level-rank duality of the coefficients of the open-string partition function of ω_c-twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$. That is, if the spectrum of an $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ open string stretched between ω_c-untwisted D-branes α and β contains $n_{\beta\lambda}{}^{\alpha}$ copies of the highest-weight representation V_λ of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$, then the spectrum of an $\widehat{\mathrm{su}}(2k+1)_{2n+1}$ open string stretched between ω_c-twisted D-branes $\tilde{\alpha}$ and $\tilde{\beta}$ contains an equal number of copies of the highest-weight representation $V_{\tilde{\lambda}}$ of $\widehat{\mathrm{su}}(2k+1)_{2n+1}$.

Proof of eq. (6.11): Let ρ, a self-conjugate representation of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$, have Dynkin indices

$$\rho = (2k+1-2\ell_1, a_1, \cdots, a_n, a_n, \cdots, a_1) \tag{6.14}$$

where $\ell_1 = \sum_{i=1}^n a_i$. The Young tableau for ρ has $r(\rho) = (2n+1)\ell_1$ boxes. The representation $\pi(\rho)'$ of $\widehat{\mathrm{sp}}(n)_k$ that corresponds to ρ has Dynkin indices $(k-\ell_1, a_1, \cdots, a_n)$. Let the transpose representation $\widetilde{\pi(\rho)}'$ of $\widehat{\mathrm{sp}}(k)_n$ have Dynkin indices $(n - \tilde{\ell}_1, \tilde{a}_1, \cdots, \tilde{a}_k)$, with $\tilde{\ell}_1 = \sum_{i=1}^k \tilde{a}_i$. The representation $\hat{\rho}$ of $\widehat{\mathrm{su}}(2k+1)_{2n+1}$ that corresponds to $\widetilde{\pi(\rho)}'$ has Dynkin indices $(2n + 1 - 2\tilde{\ell}_1, \tilde{a}_1, \cdots, \tilde{a}_k, \tilde{a}_k, \cdots, \tilde{a}_1)$. Finally, the representation $\sigma^{\ell_1}(\hat{\rho})$ has Dynkin indices

$$\sigma^{\ell_1}(\hat{\rho}) = (\tilde{a}_{\ell_1}, \tilde{a}_{\ell_1 - 1}, \cdots, \tilde{a}_1, 2n+1-2\tilde{\ell}_1, \tilde{a}_1, \cdots, \tilde{a}_{\ell_1}, 0, \cdots, 0) \tag{6.15}$$

where the last $2(k - \ell_1)$ entries vanish since $\tilde{a}_i = 0$ for $i > \ell_1$.

Since $\pi(\rho)'$ and $\widetilde{\pi(\rho)}'$ are transpose representations, with row lengths $\ell_i = \sum_{j=i}^n a_j$ and $\tilde{\ell}_i = \sum_{j=i}^k \tilde{a}_j$ respectively, their index sets, defined by [35, 36]

$$I = \{\ell_i - i + n + 1 \mid 1 \le i \le n\}, \qquad \overline{I} = \{n + i - \tilde{\ell}_i \mid 1 \le i \le \ell_1\} \tag{6.16}$$

satisfy

$$I \cup \overline{I} = \{1, 2, \cdots, n + \ell_1\}, \qquad I \cap \overline{I} = 0 \tag{6.17}$$

where we have used $\tilde{\ell}_i = 0$ for $i > \ell_1$.

To prove that the Young tableau of $\sigma^{\ell_1}(\hat{\rho})$ is the transpose of ρ, we must show that the index sets [35, 36]

$$J = \{\lambda_i - i + 2n + 2 \mid 1 \le i \le 2n+1\}, \qquad \overline{J} = \{2n + 1 + i - \hat{\lambda}_i \mid 1 \le i \le 2k+1\} \tag{6.18}$$

(where λ_i and $\hat{\lambda}_i$ are the row lengths of ρ and $\sigma^{\ell_1}(\hat{\rho})$ respectively, and $\lambda_{2n+1} = \hat{\lambda}_{2k+1} = 0$) satisfy

$$J \cup \overline{J} = \{1, 2, \cdots, 2n + 2k + 2\}, \qquad J \cap \overline{J} = 0\,. \tag{6.19}$$

Using eqs. (6.14) and (6.15), one gets

$$\begin{aligned}
J &= J_1 \cup J_2 \cup J_3, & \overline{J} &= \overline{J}_1 \cup \overline{J}_2 \cup \overline{J}_3, \\
J_1 &= \{\ell_1 + i - \ell_i \mid 1 \le i \le n\}, & \overline{J}_1 &= \{\tilde{\ell}_i - i + \ell_1 + 1 \mid 1 \le i \le \ell_1\}, \\
J_2 &= \{n + \ell_1 + 1\}, & \overline{J}_2 &= \{2n + \ell_1 + 1 + i - \tilde{\ell}_i \mid 1 \le i \le \ell_1\}, \\
J_3 &= \{2n + 2 + \ell_1 + \ell_i - i \mid 1 \le i \le n\}, & \overline{J}_3 &= \{2n + 2\ell_1 + 1 + i \mid 1 \le i \le 2k - 2\ell_1 + 1\},
\end{aligned} \tag{6.20}$$

where ℓ_i and $\tilde{\ell}_i$ are the row lengths of the $\widehat{\mathrm{sp}}(n)_k$ and $\widehat{\mathrm{sp}}(k)_n$ representations $\pi(\rho)'$ and $\widetilde{\pi(\rho)}'$. Using eq. (6.17), one observes that

$$\begin{aligned}
J_1 \cup \overline{J}_1 &= \{1, 2, \cdots, n + \ell_1\}, & J_1 \cap \overline{J}_1 &= 0, \\
J_2 &= \{n + \ell_1 + 1\}, \\
J_3 \cup \overline{J}_2 &= \{n + \ell_1 + 2, \cdots, 2n + 2\ell_1 + 1\}, & J_3 \cap \overline{J}_2 &= 0, \\
\overline{J}_3 &= \{2n + 2\ell_1 + 2, \cdots, 2n + 2k + 2\},
\end{aligned} \tag{6.21}$$

which establishes eq. (6.19). *QED.*

7 Conclusions

In this paper, we have continued our analysis, begun in ref. [38], of level-rank duality in boundary WZW models. We examined the relation between the D0-brane charges of level-rank dual untwisted D-branes of $\widehat{\mathrm{su}}(N)_K$ and $\widehat{\mathrm{sp}}(n)_k$, and of level-rank dual ω_c-twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$. We also demonstrated the level-rank duality of the spectrum of an open string stretched between untwisted or ω_c-twisted D-branes in each of these theories. The analysis of level-rank duality of ω_c-twisted D-branes of $\widehat{\mathrm{su}}(2n+1)_{2k+1}$ is facilitated by their close relation to untwisted D-branes of $\widehat{\mathrm{sp}}(n)_k$.

It is expected that level-rank duality will also be present in the boundary WZW models and D-branes of other level-rank dual groups. Also, the level-rank duality of bulk $\widehat{\mathrm{su}}(N)_K$ WZW models presumably has consequences for the twisted D-branes of boundary $\widehat{\mathrm{su}}(N)_K$ models even when N and K are not odd. The level-rank map between the twisted D-branes in these cases is expected, however, to be more complicated than for $\widehat{\mathrm{su}}(2n+1)_{2k+1}$. We leave this to future work.

Further, it would be interesting to derive the level-rank dualities described in this paper directly from K-theory.

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References

[1] C. Klimcik and P. Severa, "Open strings and D-branes in WZNW models." Nucl. Phys. **B488** (1997) 653–676, `hep-th/9609112`; M. Kato and T. Okada, "D-branes on group manifolds." Nucl. Phys. **B499** (1997) 583–595, `hep-th/9612148`; A. Y. Alekseev and V. Schomerus, "D-branes in the WZW model." Phys. Rev. **D60** (1999) 061901, `hep-th/9812193`; K. Gawedzki, "Conformal field theory: A case study." `hep-th/9904145`.

[2] R. E. Behrend, P. A. Pearce, V. B. Petkova, and J.-B. Zuber, "On the classification of bulk and boundary conformal field theories." Phys. Lett. **B444** (1998) 163–166, `hep-th/9809097`; "Boundary conditions in rational conformal field theories." Nucl. Phys. **B570** (2000) 525–589, `hep-th/9908036`.

[3] J. Fuchs and C. Schweigert, "Symmetry breaking boundaries. I: General theory." Nucl. Phys. **B558** (1999) 419–483, `hep-th/9902132`.

[4] L. Birke, J. Fuchs, and C. Schweigert, "Symmetry breaking boundary conditions and WZW orbifolds." Adv. Theor. Math. Phys. **3** (1999) 671–726, `hep-th/9905038`.

[5] A. Y. Alekseev, A. Recknagel, and V. Schomerus, "Non-commutative world-volume geometries: Branes on SU(2) and fuzzy spheres." JHEP **09** (1999) 023, `hep-th/9908040`; "Brane dynamics in background fluxes and non-commutative geometry." JHEP **05** (2000) 010, `hep-th/0003187`.

[6] G. Felder, J. Frohlich, J. Fuchs, and C. Schweigert, "The geometry of WZW branes." J. Geom. Phys. **34** (2000) 162–190, `hep-th/9909030`.

[7] S. Stanciu, "D-branes in group manifolds." JHEP **01** (2000) 025, `hep-th/9909163`; "A note on D-branes in group manifolds: Flux quantization and D0-charge." JHEP **10** (2000) 015, `hep-th/0006145`; "An illustrated guide to D-branes in SU(3)." `hep-th/0111221`; J. M. Figueroa-O'Farrill and S. Stanciu, "D-brane charge, flux quantization and relative (co)homology." JHEP **01** (2001) 006, `hep-th/0008038`.

[8] C. Bachas, M. R. Douglas, and C. Schweigert, "Flux stabilization of D-branes." JHEP **05** (2000) 048, `hep-th/0003037`; J. Pawelczyk, "SU(2) WZW D-branes and their noncommutative geometry from DBI action." JHEP **08** (2000) 006, `hep-th/0003057`; W. Taylor, "D2-branes in B fields." JHEP **07** (2000) 039, `hep-th/0004141`.

[9] A. Alekseev and V. Schomerus, "RR charges of D2-branes in the WZW model." `hep-th/0007096`; A. Y. Alekseev, A. Recknagel, and V. Schomerus, "Open strings and non-commutative geometry of branes on group manifolds." Mod. Phys. Lett. **A16** (2001) 325–336, `hep-th/0104054`.

[10] S. Fredenhagen and V. Schomerus, "Branes on group manifolds, gluon condensates, and twisted K-theory." JHEP **04** (2001) 007, `hep-th/0012164`.

[11] J. M. Maldacena, G. W. Moore, and N. Seiberg, "Geometrical interpretation of D-branes in gauged WZW models." JHEP **07** (2001) 046, `hep-th/0105038`.

[12] J. M. Maldacena, G. W. Moore, and N. Seiberg, "D-brane instantons and K-theory charges." JHEP **11** (2001) 062, hep-th/0108100.

[13] K. Gawedzki, "Boundary WZW, G/H, G/G and CS theories." Annales Henri Poincare **3** (2002) 847–881, hep-th/0108044; K. Gawedzki and N. Reis, "WZW branes and gerbes." Rev. Math. Phys. **14** (2002) 1281–1334, hep-th/0205233.

[14] H. Ishikawa, "Boundary states in coset conformal field theories." Nucl. Phys. **B629** (2002) 209–232, hep-th/0111230.

[15] V. B. Petkova and J. B. Zuber, "Boundary conditions in charge conjugate sl(N) WZW theories." hep-th/0201239.

[16] M. R. Gaberdiel and T. Gannon, "Boundary states for WZW models." Nucl. Phys. **B639** (2002) 471–501, hep-th/0202067.

[17] M. R. Gaberdiel and T. Gannon, "The charges of a twisted brane." JHEP **01** (2004) 018, hep-th/0311242.

[18] M. R. Gaberdiel, T. Gannon, and D. Roggenkamp, "The D-branes of SU(n)." JHEP **07** (2004) 015, hep-th/0403271; M. R. Gaberdiel, T. Gannon, and D. Roggenkamp, "The coset D-branes of SU(n)." JHEP **10** (2004) 047, hep-th/0404112.

[19] A. Y. Alekseev, S. Fredenhagen, T. Quella, and V. Schomerus, "Non-commutative gauge theory of twisted D-branes." Nucl. Phys. **B646** (2002) 127–157, hep-th/0205123; T. Quella, "Branching rules of semi-simple Lie algebras using affine extensions." J. Phys. **A35** (2002) 3743–3754, math-ph/0111020.

[20] T. Quella and V. Schomerus, "Symmetry breaking boundary states and defect lines." JHEP **06** (2002) 028, hep-th/0203161; T. Quella, "On the hierarchy of symmetry breaking D-branes in group manifolds." JHEP **12** (2002) 009, hep-th/0209157.

[21] P. Bouwknegt, P. Dawson, and D. Ridout, "D-branes on group manifolds and fusion rings." JHEP **12** (2002) 065, hep-th/0210302.

[22] P. Bouwknegt and D. Ridout, "A note on the equality of algebraic and geometric D-brane charges in WZW models." JHEP **05** (2004) 029, hep-th/0312259.

[23] H. Ishikawa and T. Tani, "Novel construction of boundary states in coset conformal field theories." Nucl. Phys. **B649** (2003) 205–242, hep-th/0207177; "Twisted boundary states and representation of generalized fusion algebra." hep-th/0510242; H. Ishikawa and A. Yamaguchi, "Twisted boundary states in c = 1 coset conformal field theories." JHEP **04** (2003) 026, hep-th/0301040.

[24] H. Ishikawa and T. Tani, "Twisted boundary states in Kazama-Suzuki models." Nucl. Phys. **B678** (2004) 363–397, hep-th/0306227; S. Schafer-Nameki, "D-branes in N = 2 coset models and twisted equivariant K- theory." hep-th/0308058;

[25] V. Braun and S. Schafer-Nameki, "Supersymmetric WZW models and twisted K-theory of SO(3)." `hep-th/0403287`.

[26] M. R. Gaberdiel and T. Gannon, "D-brane charges on non-simply connected groups." JHEP **04** (2004) 030, `hep-th/0403011`; S. Fredenhagen, "D-brane charges on SO(3)." JHEP **11** (2004) 082, `hep-th/0404017`; T. Gannon and M. Vasudevan, "Charges of exceptionally twisted branes." JHEP **07** (2005) 035, `hep-th/0504006`; S. Fredenhagen, M. R. Gaberdiel, and T. Mettler, "Charges of twisted branes: The exceptional cases." JHEP **05** (2005) 058, `hep-th/0504007`; S. Fredenhagen and T. Quella, "Generalised permutation branes." JHEP **11** (2005) 004, `hep-th/0509153`.

[27] V. Schomerus, "Lectures on branes in curved backgrounds." Class. Quant. Grav. **19** (2002) 5781–5847, `hep-th/0209241`.

[28] M. B. Halpern and C. Helfgott, "A basic class of twisted open WZW strings." Int. J. Mod. Phys. **A19** (2004) 3481–3540, `hep-th/0402108`; "The general twisted open WZW string." Int. J. Mod. Phys. **A20** (2005) 923–992, `hep-th/0406003`.

[29] J. L. Cardy, "Boundary conditions, fusion rules and the Verlinde formula." Nucl. Phys. **B324** (1989) 581.

[30] R. Minasian and G. W. Moore, "K-theory and Ramond-Ramond charge." JHEP **11** (1997) 002, `hep-th/9710230`; E. Witten, "D-branes and K-theory." JHEP **12** (1998) 019, `hep-th/9810188`; G. W. Moore, "K-theory from a physical perspective." `hep-th/0304018`.

[31] P. Bouwknegt and V. Mathai, "D-branes, B-fields and twisted K-theory." JHEP **03** (2000) 007, `hep-th/0002023`.

[32] D. S. Freed, "The Verlinde algebra is twisted equivariant K-theory." Turk. J. Math. **25** (2001) 159–167, `math.rt/0101038`; "Twisted K-theory and loop groups." `math.at/0206237`; D. S. Freed, M. J. Hopkins, and C. Teleman, "Twisted K-theory and Loop Group Representations." `math.at/0312155`.

[33] V. Braun, "Twisted K-theory of Lie groups." JHEP **03** (2004) 029, `hep-th/0305178`; C. L. Douglas, "On the Twisted K-Homology of Simple Lie Groups." `math.at/0404017`.

[34] S. G. Naculich and H. J. Schnitzer, "Duality between SU(N)-k and SU(k)-N WZW models." Nucl. Phys. **B347** (1990) 687–742; "Duality relations between SU(N)-k and SU(k)-N WZW models and their braid matrices." Phys. Lett. **B244** (1990) 235–240; S. G. Naculich, H. A. Riggs, and H. J. Schnitzer, "Group level duality in WZW models and Chern-Simons theory." Phys. Lett. **B246** (1990) 417–422; J. Fuchs and P. van Driel, "Some symmetries of quantum dimensions." J. Math. Phys. **31** (1990) 1770–1775.

[35] D. Altschuler, M. Bauer, and C. Itzykson, "The branching rules of conformal embeddings." Commun. Math. Phys. **132** (1990) 349–364 (see also M. A. Walton,

"Conformal branching rules and modular invariants." Nucl. Phys. **B322** (1989) 775); H. Saleur and D. Altschuler, "Level rank duality in quantum groups." Nucl. Phys. **B354** (1991) 579–613; A. Kuniba and T. Nakanishi, "Level rank duality in fusion RSOS models." In *Proceedings of the International Colloquium on Modern Quantum Field Theory*, Bombay, January 1990 (World Scientific, Singapore, 1991).

[36] E. J. Mlawer, S. G. Naculich, H. A. Riggs, and H. J. Schnitzer, "Group level duality of WZW fusion coefficients and Chern- Simons link observables." Nucl. Phys. **B352** (1991) 863–896.

[37] M. Bourdeau, E. J. Mlawer, H. Riggs, and H. J. Schnitzer, "Topological Landau-Ginzburg matter from Sp(N)-K fusion rings." Mod. Phys. Lett. **A7** (1992) 689–700, `hep-th/9111020`.

[38] S. G. Naculich and H. J. Schnitzer, "Level-rank duality of D-branes on the SU(N) group manifold." Nucl. Phys. **B**, to appear, `hep-th/0511083`.

[39] N. Ishibashi, "The boundary and crosscap states in conformal field theories." Mod. Phys. Lett. **A4** (1989) 251.

[40] E. P. Verlinde, "Fusion rules and modular transformations in 2-d conformal field theory." Nucl. Phys. **B300** (1988) 360.

[41] I. Affleck and A. W. W. Ludwig, "The Kondo effect, conformal field theory and fusion rules." Nucl. Phys. **B352** (1991) 849–862; "Critical theory of overscreened Kondo fixed points." Nucl. Phys. **B360** (1991) 641–696.

[42] J. Fuchs, B. Schellekens, and C. Schweigert, "From Dynkin diagram symmetries to fixed point structures." Commun. Math. Phys. **180** (1996) 39–98, `hep-th/9506135`.

[43] P. Goddard and D. I. Olive, "Kac-Moody and Virasoro algebras in relation to quantum physics." Int. J. Mod. Phys. **A1** (1986) 303.

[44] W. Fulton and J. Harris, *Representation Theory* (Springer-Verlag, 1991)

[45] J. Fuchs, *Affine Lie Algebras and Quantum Groups* (Cambridge University Press, 1992)

[46] V. Kac, *Infinite dimensional Lie algebras*, 3rd ed., (Cambridge University Press, 1990)